Available online at www.sciencedirect.com
ScienceDirect
Journal of Geometry and Physics 57 (2006) 89–100
JOURNAL OF GEOMETRY AND PHYSICS
www.elsevier.com/locate/jgp

# On the non-integrability of a generalized Darboux Halphen system

Clàudia Valls

*Departamento de Matemática, Instituto Superior Técnico, Av. Rovisco Pais 1049-001, Lisboa, Portugal*

Received 28 October 2005; received in revised form 6 February 2006; accepted 9 February 2006
Available online 20 March 2006

**Abstract**

In this paper we study a generalized Darboux Halphen system given by

$$\begin{aligned}\dot{x}_1 &= x_2x_3 - x_1(x_2+x_3) + \tau^2(\alpha_1,\alpha_2,\alpha_3,x_1,x_2,x_3),\\ \dot{x}_2 &= x_3x_1 - x_2(x_3+x_1) + \tau^2(\alpha_1,\alpha_2,\alpha_3,x_1,x_2,x_3),\\ \dot{x}_3 &= x_1x_2 - x_3(x_1+x_2) + \tau^2(\alpha_1,\alpha_2,\alpha_3,x_1,x_2,x_3),\end{aligned}$$

where $x_1, x_2, x_3$ are real variables, $\alpha_1, \alpha_2, \alpha_3$ are real constants and

$$\tau^2(\alpha_1,\alpha_2,\alpha_3,x_1,x_2,x_3) = \alpha_1^2(x_1-x_2)(x_3-x_1) + \alpha_2^2(x_2-x_3)(x_1-x_2) + \alpha_3^2(x_3-x_1)(x_2-x_3).$$

We prove that, for any $(\alpha_1,\alpha_2,\alpha_3) \in \mathbb{R}^3 \setminus \{(0,0,0)\}$, this system does not admit any non-constant global first integral that can be described by a formal power series. Furthermore, restricting the values of $(\alpha_1,\alpha_2,\alpha_3)$ to a full Lebesgue measure set, we prove that this system does not admit any non-constant rational or Darbouxian global first integral. This is a first step toward proving that this system is chaotic.


*MSC:* 34C35; 34D30

*Keywords:* Generalized Halphen system; Analytic and rational first integrals

## 1. Introduction to the problem

In this paper we consider the system

$$\begin{aligned}\dot{x}_1 &:= F_1(x_1,x_2,x_3) = x_2x_3 - x_1(x_2+x_3) + \tau^2(\alpha_1,\alpha_2,\alpha_3,x_1,x_2,x_3),\\ \dot{x}_2 &:= F_2(x_1,x_2,x_3) = x_3x_1 - x_2(x_3+x_1) + \tau^2(\alpha_1,\alpha_2,\alpha_3,x_1,x_2,x_3),\\ \dot{x}_3 &:= F_3(x_1,x_2,x_3) = x_1x_2 - x_3(x_1+x_2) + \tau^2(\alpha_1,\alpha_2,\alpha_3,x_1,x_2,x_3),\end{aligned} \tag{1}$$

where $x_1, x_2, x_3$ are real variables, $\alpha_1, \alpha_2, \alpha_3$ are real constants and

$$\tau^2(\alpha_1,\alpha_2,\alpha_3,x_1,x_2,x_3) = \alpha_1^2(x_1-x_2)(x_3-x_1) + \alpha_2^2(x_2-x_3)(x_1-x_2) + \alpha_3^2(x_3-x_1)(x_2-x_3).$$

*E-mail address:* cvalls@math.ist.utl.pt.

0393-0440/$ - see front matter © 2006 Elsevier B.V. All rights reserved.
doi:10.1016/j.geomphys.2006.02.004

We will refer to system (1) as the generalized Darboux–Halphen system since, when $\tau = 0$, i.e., $\alpha_1 = \alpha_2 = \alpha_3 = 0$, system (1) becomes the classical Halphen system which first appeared in Darboux's work (see [12]). System (1) arises in the study of SU(2)-invariant hypercomplex manifolds (see [17]) and is a reduction of the self-dual Yang-Mills equations corresponding to an infinite-dimensional gauge group of diffeomorphism Diff$(S^3)$ of a three-sphere (see [6]). Furthermore, it describes a class of self-dual Weyl Bianchi IX space-times with Euclidean signature (see [5]). Special cases of (1) arise in the study of solvable models of spherically symmetric shear-free fluids in general relativity (see [20]).

The question to be settled is the generic behavior of system (1): Is it chaotic or not? Many numerical computations have been performed in this direction (see [2,3,14,18,21,22]) which seems to indicate a probable chaotic behavior for this system. It is well known that the existence of complicated behavior of a system forbids its integrability and thus, as a first step toward understanding the underlying mechanism for chaotic behavior of system (1), we will prove analytically that, for almost all values of the parameters $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$, system (1) does not admit any non-constant global first integral that is neither a formal power series, nor a rational function, nor a Darbouxian function. We emphasize that this is a first step toward proving that this system is chaotic. However, those results do not allow us to conclude that the system is chaotic. To settle the question, one should go further and prove that the system has a positive Lyapunov exponent, or has homoclinic/heteroclinic connections, or it is not analytically integrable.

Halphen showed in [16] that system (1) can be solved in terms of hypergeometric functions. Special solutions have also been given in terms of theta functions and automorphic forms (see [1,15]). We want to point out that in [7] the authors prove that, indeed, system (1) can be integrated explicitly, since they can express its general solution in terms of transcendental and non-meromorphic functions. However, using [23] it is clear that these first integrals are not global and are multi-valued non-algebraic functions. Therefore, the existence of those first integrals tells us nothing about the presence or absence of chaotic behavior.

When $\tau = 0$, system (1) has been intensively studied from the point of view of integrability and it has been proved that it does not admit any non-constant first integral that can be described either by a rational function or by a formal power series (see [24] for further references). The aim of this paper is then to study the existence of global first integrals of system (1) when $\tau \neq 0$. To do this, we will mainly use the theory of integrability of Darboux (see [13,19]). This theory goes back to Darboux, who showed in [13] how to construct the first integrals of a polynomial system $\dot{x} = g(x)$. If we can determine polynomials $f$ and $K$ such that $\nabla f \cdot \dot{x} = K f$, then the equation $f = 0$ describes a surface formed by trajectories; it is an algebraic solution of the system. Finally, one looks for the first integrals as products of a sufficient number of algebraic functions $f$ raised to a given power. The application of this theory strongly depends on the desired results, and appropriate and additional techniques may need to be developed in each particular case. We want to mention [4,8,9,19,25,26], among others, for further applications of this method in different problems. In this paper, we apply the Darboux theory of integrability together with the following idea. The three hyperplanes

$$H_1 := x_1 - x_2 = 0, \quad H_2 := x_1 - x_3 = 0 \quad \text{and} \quad H_3 := x_2 - x_3 = 0 \tag{2}$$

are invariant by the flow of system (1) and, if $f := f(x_1, x_2, x_3)$ is a first integral of system (1), then for each $i = 1, 2, 3$ the restriction of $f$ to $H_i = 0$ is also a first integral of system (1) restricted to $H_i = 0$. Thus the method of proof will consist of studying completely the integrability of the reduced system (1) on each $H_i = 0$ to get exact information on the integrals of the whole system (1).

A *formal first integral* $f = f(x_1, x_2, x_3)$ of system (1) is a formal power series in the variables $x_1, x_2, x_3$ such that $\sum_{k=1}^{3} \frac{\partial f}{\partial x_k} F_k(x_1, x_2, x_3) = 0$.

The first main result of this paper is:

**Theorem 1.** *For every* $(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3 \setminus \{(0, 0, 0)\}$, *system* (1) *does not admit non-constant formal first integrals.*

As a corollary we readily obtain the following result:

**Theorem 2.** *For every* $(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3 \setminus \{(0, 0, 0)\}$, *system* (1) *does not admit non-constant polynomial first integrals.*

Here, an *analytic first integral* of system (1) is an analytic function that is constant over the trajectories of system (1) and is different from a polynomial. The second main result of this paper is:

**Theorem 3.** *For every* $(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3 \setminus \{(0, 0, 0)\}$*, system* (1) *does not have analytic first integrals in a neighborhood of the origin.*

To introduce the two remaining main theorems in this paper, we introduce the following full-Lebesgue measure set, $\Gamma$, of values of $\alpha$:

$$\Gamma := \{(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3 \setminus \{(0, 0, 0)\} : \text{ for all } (n_1, n_2, n_3) \in \mathbb{Z}^3 \setminus \{(0, 0, 0)\}, \\ n_1\alpha_1 + n_2\alpha_2 + n_3\alpha_3 \notin \mathbb{Z}^- \setminus \{0\}\}. \tag{3}$$

A *rational first integral* $f = f(x_1, x_2, x_3)$ of system (1) is a rational function that is constant over the solutions of system (1). The third main result of this paper is the following.

**Theorem 4.** *For* $(\alpha_1, \alpha_2, \alpha_3) \in \Gamma$*, system* (1) *does not admit non-constant rational first integrals.*

Finally, the last main result of this paper is concerned with the Darbouxian first integrals (see below for their definition).

**Theorem 5.** *For* $(\alpha_1, \alpha_2, \alpha_3) \in \Gamma$*, system* (1) *does not admit non-constant Darbouxian first integrals.*

This paper is organized as follows. In Section 2, we introduce some generalities that will lead to the proof of all the main theorems. In Section 3, we will prove Theorems 1 and 3. In Section 4, we will prove Theorem 4 and finally, in Section 5, we will prove Theorem 5.

## 2. Generalities

Consider a polynomial system

$$\dot{x}_j = X_j(x_1, \dots, x_n), \quad j = 1, \dots, n. \tag{4}$$

We say that a non-constant polynomial $f \in \mathbb{C}[x_1, \dots, x_n]$ is a *Darboux polynomial* of system (4) if there exists $K \in \mathbb{C}[x_1, \dots, x_n]$, called a *cofactor* of $f$, such that $\sum_{j=1}^n X_j \frac{\partial f}{\partial x_j} = Kf$. Note that if $(X_1, \dots, X_n)$ has degree $k$, then $K$ has degree at most $k - 1$. Furthermore, $\frac{\mathrm{d}}{\mathrm{d}t} f(x(t)) = K(x(t)) f(x(t))$, where $x(t) = (x_1(t), \dots, x_n(t))$ is a solution of system (4). Therefore, $f = 0$ is an invariant algebraic hypersurface for the flow of system (4), and a polynomial first integral of system (4) is a Darboux polynomial with cofactor zero. If there exist invariant hyperplanes under the flow of (4) and $f$ is a Darboux polynomial of (4) with cofactor $K$, then the restriction of $f$ to each of the invariant hyperplanes is a Darboux polynomial of system (4) restricted to each of the hyperplanes and with cofactors being the restriction of the cofactor $K$ to each of the invariant hyperplanes. We note that, for real polynomial differential systems, such as system (4), when we look for their Darbouxian first integrals we use, in general, complex Darboux polynomials and complex exponential factors. This is due to the fact that these objects appear in pairs (it and its conjugate) and this forces the Darbouxian first integral to become real. For more details, see [11]. It is simple to show the following lemma.

**Lemma 6.** *If we decompose the polynomial* $f$ *into its irreducible factors in* $\mathbb{C}[x_1, \dots, x_n]$ *as* $\prod_{j=1}^s f_j^{n_j}$, $n_j \in \mathbb{N} \cup \{0\}$, *then* $f$ *is a Darboux polynomial if and only if every* $f_j$ *is a Darboux polynomial. Moreover, if* $K$ *and* $K_j$ *are the cofactors of* $f$ *and* $f_j$*, then* $K = \sum_{j=1}^s n_j K_j$.

The following statement is crucial to investigate the rational integrability of a polynomial system. It can be proved easily, and thus its proof if not included.

**Proposition 7.** *The existence of a rational first integral for a polynomial ordinary differential equation* (4) *implies either: the existence of a polynomial first integral (and thus a Darboux polynomial with zero cofactor) or the existence of two coprime Darboux polynomials with the same non-zero cofactor.*

An *exponential factor* $F$ of the polynomial differential system (4) is a function $F = \exp(f/g) \notin \mathbb{C}$ with $f, g \in \mathbb{C}[x_1, \dots, x_n]$, coprime and satisfying that $\sum_{j=1}^n X_j \frac{\partial F}{\partial x_j} = LF$, for some polynomial $L \in \mathbb{C}[x_1, \dots, x_n]$. Note that if $(X_1, \dots, X_n)$ has degree $k$, then $L$ has degree at most $k - 1$. The proof of the following two results can be found in [10] and [11].

**Proposition 8.** *If $F = \exp(h/g)$ is an exponential factor for the polynomial differential system (4) and $g$ is not a constant polynomial, then $g = 0$ is an invariant algebraic hypersurface of system (4) with multiplicity higher than* 1.

A first integral $G$ of system (4) is called *Darboux* if $G = f_1^{\lambda_1} \dots f_p^{\lambda_p} F_1^{\mu_1} \dots F_q^{\mu_q}$, where $f_1, \dots, f_p$ are Darboux polynomials, $F_1, \dots, F_q$ are exponential factors, and $\lambda_j, \mu_k \in \mathbb{C}$, for $j = 1, \dots, p$, $k = 1, \dots, q$.

**Theorem 9.** *Suppose that the differential polynomial system (4) defined in $\mathbb{R}^n$ of degree $m$ admits $p$ invariant algebraic hypersurfaces $f_i = 0$ with cofactors $K_i$ for $i = 1, \dots, p$ and $q$ exponential factors $F_j = \exp(g_j/h_j)$ with cofactors $L_j$ for $j = 1, \dots, q$. Then, there exist $\lambda_j, \mu_j \in \mathbb{C}$ not all zero such that $\sum_{i=1}^{p} \lambda_i K_i + \sum_{j=1}^{q} \mu_j L_j = 0$ if and only if the following real (multi-valued) function of Darboux type $f_1^{\lambda_1} \dots f_p^{\lambda_p} F_1^{\mu_1} \dots F_q^{\mu_q}$, substituting $f_i^{\lambda_i}$ by $|f_i|^{\lambda_i}$ if $\lambda_i \in \mathbb{R}$, is a first integral of system (4).*

## 3. Proof of Theorems 1 and 3

In this section, we prove Theorems 1 and 3. For this, we first introduce some preliminary results that will be used in the proofs. The next result can be proved easily using Newton's binomial formula, and thus its proof has been omitted.

**Lemma 10.** *Let $f = f(x_1, x_2, x_3)$ be a formal power series such that in $x_l = x_j$, $l, j \in \{1, 2, 3\}$, $l \neq j$, we have $f(x_1, x_2, x_3)|_{x_l = x_j} = \overline{f}$, where $\overline{f}$ is a formal power series in the variables $x_j, x_k$ with $k \in \{1, 2, 3\}$, $k \neq j$ and $k \neq l$. Then, there exists a formal series $g = g(x_1, x_2, x_3)$ such that $f = \overline{f} + (x_l - x_j)g$.*

As pointed out in the introduction, the hyperplanes $\{x_1 = x_2\}$, $\{x_1 = x_3\}$ and $\{x_2 = x_3\}$ are invariant under the flow of (1). Therefore, if $f$ is a formal first integral of system (1), then

$$f_1(x_2, x_3) = f(x_2, x_2, x_3), \quad f_2(x_2, x_3) = f(x_3, x_2, x_3), \quad f_3(x_1, x_3) = f(x_1, x_3, x_3), \tag{5}$$

are formal first integrals of system (1) restricted to the hyperplanes $\{x_1 = x_2\}$, $\{x_1 = x_3\}$ and $\{x_2 = x_3\}$.

**Proposition 11.** *For $(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3 \setminus \{(0, 0, 0)\}$, if $f$ is a formal first integral of system (1), then*

$$f = c_0 + (x_1 - x_2)(x_1 - x_3)(x_2 - x_3)g,$$

*where $c_0$ is some constant and $g := g(x_1, x_2, x_3)$ is a formal power series.*

**Proof.** Let $f$ be a formal first integral of system (1). We will first prove that $f_1 = c_0$. Indeed, $f_1$ satisfies $-(x_2^2 + \alpha_3^2(x_2 - x_3)^2)\frac{\partial f_1}{\partial x_2} + (x_2^2 - 2x_2x_3 - \alpha_3^2(x_2 - x_3)^2)\frac{\partial f_1}{\partial x_3} = 0$. We now introduce the linear change of variables

$$y_2 = x_2, \quad z_2 = x_2 - x_3. \tag{6}$$

In the new variables, we have $f_1(x_2, x_3) = h(y_2, z_2)$ and $h$ satisfies

$$-(y_2^2 + \alpha_3^2 z_2^2)\frac{\partial h}{\partial y_2} - 2y_2z_2\frac{\partial h}{\partial z_2} = 0. \tag{7}$$

We want to prove that $h = c_0$. For this, we write $h$ in power series in the variables $y_2$ and $z_2$ as $h = \sum_{k,l \geq 0} h_{k,l} y_2^k z_2^l$. Thus, imposing that $h$ satisfies (7), we get that

$$\begin{aligned} 0 &= \sum_{k,l \geq 0} (k + 2l)h_{k,l} y_2^{k+1} z_2^l + \alpha_3^2 \sum_{k,l \geq 0} k h_{k,l} y_2^{k-1} z_2^{l+1} \\ &= \sum_{k,l \geq 0} \left((k + 2l - 1)h_{k-1,l} + \alpha_3^2(k + 1)h_{k+1,l-1}\right) y_2^k z_2^l, \end{aligned} \tag{8}$$

where $h_{m,n} = 0$ for $m < 0$ or $n < 0$. Now, computing the different degrees in the variables $y_2$ and $z_2$ in (8), we get that, for $k, l \geq 0$,

$$(k + 2l)h_{k,l} + \alpha_3^2(k + 2)h_{k+2,l-1} = 0, \tag{9}$$

where $h_{m,n} = 0$ for $n < 0$. We claim that

$$h_{k,l} = 0 \quad \text{for } k, l \geq 0, (k, l) \neq (0, 0). \tag{10}$$

We will prove (10) by induction over $l$. For $l = 0$, (9) implies that $kh_{k,0} = 0$ for all $k \geq 0$, which clearly yields $h_{k,0} = 0$ for all $k > 0$ and finishes the proof of (10) for $l = 0$. Now, assume that (10) is true for $k = 0, \ldots, m-1$ $(m \geq 1)$ and we will prove it for $k = m$. By inductive hypothesis, $h_{k,m}$ satisfies $(k+2m)h_{k,m} = 0$ for $k \geq 0$, which yields $h_{k,m} = 0$ for $k \geq 0$. Then, (10) is proved for $k = m$ and, by induction, (10) holds. Then, from (10) we get that $h = h_{0,0} := c_0$ and, in view of (6), we obtain $f_1(x_2, x_3) = c_0$. Then, using Lemma 10 with $x_l = x_1$ and $x_j = x_2$, we obtain

$$f = c_0 + (x_1 - x_2)g_0, \tag{11}$$

for some formal power series $g_0 := g_0(x_1, x_2, x_3)$.

Now, repeating for $f_2$ the arguments we made for $f_1$, we get that there exists a positive constant $c_1$ and a formal power series $g_1 := g_1(x_1, x_2, x_3)$, such that

$$f = c_1 + (x_1 - x_3)g_1. \tag{12}$$

Finally, repeating for $f_3$ the arguments we made for $f_1$, we get that there exists a positive constant $c_2$ and a formal power series $g_2 := g_2(x_1, x_2, x_3)$, such that

$$f = c_2 + (x_2 - x_3)g_2. \tag{13}$$

Now, evaluating Eqs. (11)–(13) on $x_1 = x_2 = x_3 = 0$ and equating them, we get that $c_0 = c_1 = c_2$. Furthermore, equating (11)–(13), we get $(x_1 - x_2)g_0 = (x_1 - x_3)g_1 = (x_2 - x_3)g_2$, which clearly implies that there exists a formal power series $g := g(x_1, x_2, x_3)$ such that

$$g_0 = (x_1 - x_3)(x_2 - x_3)g, \quad g_1 = (x_1 - x_2)(x_2 - x_3)g, \quad g_2 = (x_1 - x_2)(x_1 - x_3)g. \tag{14}$$

Therefore, the proposition follows from (11) and the first relation in (14). □

**Proof of Theorem 1.** Let $f$ be any formal first integral of system (1). By Proposition 11, we know that $f$ can be written as

$$f = c_0 + (x_1 - x_2)(x_1 - x_3)(x_2 - x_3)g, \tag{15}$$

for some constant $c_0$ and some formal power series $g := g(x_1, x_2, x_3)$. Imposing that $f$ is a first integral of system (1), we get that, after simplifying by $(x_1 - x_2)(x_1 - x_3)(x_2 - x_3)$, $g$ must satisfy

$$\frac{\mathrm{d}g}{\mathrm{d}t} = 2(x_3 + x_2 + x_1)g, \tag{16}$$

where the derivative is evaluated along a solution of system (1). We will prove that $g = 0$. For this, we will proceed by reduction to the absurd. Assume that $g \neq 0$ and we will reach a contradiction. We consider two different cases.

*Case 1: $g$ is not divisible by $x_1 - x_2$.* In this case, using Lemma 10 with $x_l = x_1$ and $x_j = x_2$, we can write $g$ as $g = g_0 + (x_1 - x_2)g_1$, where $g_0 := g_0(x_2, x_3) \neq 0$ and $g_1 := g_1(x_1, x_2, x_3)$ are formal power series. Then, $g_0$ satisfies (16) restricted to $x_1 = x_2$. Thus, introducing again the change of variables (6), we have that, in the new variables, if $g_0(x_2, x_3) = h_0(y_2, z_2)$ then $h_0$ satisfies

$$-(y_2^2 + \alpha_3^2 z_2^2)\frac{\partial h_0}{\partial y_2} - 2y_2 z_2 \frac{\partial h_0}{\partial z_2} = 2(3y_2 - z_2)h_0. \tag{17}$$

Now, we write $h_0 = \sum_{j \geq 0} h_{0,j} z_2^j$, $h_j = h_j(y_2)$ and $h_j$ are formal power series for each $j$. We claim that

$$h_{0,j} = 0 \quad \text{for } j \geq 0. \tag{18}$$

Clearly, $h_{0,0}$ satisfies (17) restricted to $z_2 = 0$, that is, $-y_2^2 \frac{\mathrm{d}h_{0,0}}{\mathrm{d}y_2} = 6y_2 h_{0,0}$, i.e., $h_{0,0} = \frac{c}{y_2^6}$, where $c \in \mathbb{C}$. Since $h_{0,0}$ is a formal series in the variable $y_2$, we have that $c = 0$ and, thus, $h_{0,0} = 0$, which proves (18) for $j = 0$. Now, we assume that (18) is true for $j = 0, \dots, m-1$ ($m \geq 1$) and we will prove it for $j = m$. Clearly, by hypothesis of induction, $h_0 = \sum_{j\geq 0} h_{0,j+m} z_2^{j+m}$ and then, from (17), after dividing by $z_2^m$, we obtain

$$-(y_2^2 + \alpha_3^2 z_2^2) \sum_{j\geq 0} \frac{\mathrm{d}h_{0,j+m}}{\mathrm{d}y_2} z_2^j - 2y_2 \sum_{j\geq 0} (j+m) h_{0,j+m} z_2^j = 2(3y_2 - z_2) \sum_{j\geq 0} h_{0,j+m} z_2^j. \tag{19}$$

Then, evaluating (19) on $z_2 = 0$, we get $-y_2^2 \frac{\mathrm{d}h_{0,m}}{\mathrm{d}y_2} = y_2(6+2m) h_{0,m}$, that is, $h_{0,m} = \frac{c_m}{y_2^{6+2m}}$, where $c_m \in \mathbb{C}$. Since $h_{0,m}$ is a formal series in the variable $y_2$, we have that $c_m = 0$ and, thus, $h_{0,m} = 0$, which proves (18) for $j = m$. Then, by the induction process, (18) holds, and from (18) we get that $h_0 = 0$. Hence, using (6) we obtain that $g_0 = 0$, a contradiction.

*Case 2: $g$ is divisible by $x_1 - x_2$.* In this case, $g = (x_1 - x_2)^j G$ with $j \geq 1$, $G \neq 0$ and $G := G(x_1, x_2, x_3)$ is a formal power series such that is not divisible by $x_1 - x_2$ and satisfies, after dividing by $(x_1 - x_2)^j$ (see (16)), $\frac{\mathrm{d}G}{\mathrm{d}t} = 2(x_1 + x_2 + (j+1)x_3)G$, where the derivative of $G$ is evaluated along a solution of system (1). Then, the same arguments used for $g$ allow us to conclude that $G = 0$, a contradiction.

Hence, $g = 0$ and the proof of the theorem follows from (15). □

**Proof of Theorem 3.** To prove that system (1) does not have non-constant analytic first integrals in a neighborhood of zero, we proceed by contradiction. Assume that $g$ is a non-constant analytic first integral of system (1) in a neighborhood $U \subset \mathbb{R}^3$ of the origin. Clearly, $g_{|U}$ can be written as a formal power series which turns out to be convergent. Hence, in $U$, $g$ is a non-constant formal first integral of system (1), a contradiction with Theorem 1. Thus, Theorem 3 is proved. □

## 4. Proof of Theorem 4

We recall that the equation defining a Darboux polynomial $f$ for system (1) is $\dot{x}_1 \frac{\partial f}{\partial x_1} + \dot{x}_2 \frac{\partial f}{\partial x_2} + \dot{x}_3 \frac{\partial f}{\partial x_3} = Kf$. Furthermore, since the polynomials in the right-hand side of (1) have degree three, the cofactor $K$ has degree at most two. We write it as $K = a_0 + a_1 x_1 + a_2 x_2 + a_3 x_3$.

In the notation introduced in (5), we have that $f_1$, $f_2$ and $f_3$ are Darboux polynomials of system (1) restricted to the hyperplanes $\{x_1 = x_2\}$, $\{x_1 = x_3\}$ and $\{x_2 = x_3\}$ with cofactors $K_1$, $K_2$ and $K_3$, respectively, where $K_1$ is the restriction of $K$ to $\{x_1 = x_2\}$, $K_2$ is the restriction of $K$ to $\{x_1 = x_3\}$, and $K_3$ is the restriction of $K$ to $\{x_2 = x_3\}$.

To prove Theorem 3 with the help of Proposition 7 and Theorem 2, we will establish, in this section, the following statement.

**Theorem 12.** *For $(\alpha_1, \alpha_2, \alpha_3) \in \Gamma$, every Darboux polynomial $f$ of system (1) is of the form $f = cH_1^{n_1} H_2^{n_2} H_3^{n_3} := c(x_1 - x_2)^{n_1}(x_1 - x_3)^{n_2}(x_2 - x_3)^{n_3}$, where c is some constant and $n_1$, $n_2$, $n_3$ are non-negative integers. Furthermore, the cofactor of $f$ is $K = -2n_1 x_3 - 2n_2 x_2 - 2n_3 x_1$.*

The main objective of this section is to prove Theorem 12 since, as will be clear later, this will readily imply the proof of Theorem 4. For this, we first prove that, to study the Darboux polynomials of system (1), it is enough to consider homogeneous Darboux polynomials and the cofactor is just $K = a_1 x_1 + a_2 x_2 + a_3 x_3$. This is stated in the following two propositions. They can be proved easily and hence their proofs have been omitted.

**Proposition 13.** *For $(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3 \setminus \{(0,0,0)\}$, any Darboux polynomial $f \neq 0$ of system (1) has a cofactor of the form $K = a_1 x_1 + a_2 x_2 + a_3 x_3$.*

**Proposition 14.** *For $(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3 \setminus \{(0,0,0)\}$, if we write $f$ in the sum of its homogeneous parts as $f = f_1 + \cdots + f_n$, then $f$ is a Darboux polynomial of system (1) with cofactor $K$ if and only if, for all $j = 1, \dots, n$, $f_j$ is a Darboux polynomial of system (1) with cofactor $K$.*

In view of Proposition 13, from now on we will work with the cofactor $K$ as in Proposition 13.

Now, to prove Theorem 12 we will prove two auxiliary results. The first one deals with irreducible homogeneous Darboux polynomials of degree one and the second one deals with irreducible homogeneous Darboux polynomials of degree greater than or equal to two.

**Proposition 15.** *For* $(\alpha_1, \alpha_2, \alpha_3) \in \Gamma$*, the unique homogeneous Darboux polynomials of system* (1) *of degree one are* $H_1 := x_1 - x_2$*,* $H_2 := x_1 - x_3$ *and* $H_3 := x_2 - x_3$*, respectively, with cofactors,* $-2x_3$*,* $-2x_2$ *and* $-2x_1$*.*

**Proof.** Let $(\alpha_1, \alpha_2, \alpha_3) \in \Gamma$ (see (3)) and let $f$ be an homogeneous Darboux polynomial of system (1) of degree one. We introduce the change of variables

$$y_1 = x_1, \quad y_2 = x_1 - x_2, \quad y_3 = x_1 - x_3. \tag{20}$$

In the new variables, we have $f(x_1, x_2, x_3) = g(y_1, y_2, y_3)$ and $g := g(y_1, y_2, y_3)$ is an homogeneous polynomial of degree one. We write it as $g = b_1 y_1 + b_2 y_2 + b_3 y_3$, $(b_1, b_2, b_3) \in \mathbb{C}^3 \setminus \{(0, 0, 0)\}$. Then, $g$ is a Darboux polynomial of system

$$\begin{aligned} \dot{y}_1 &= -y_1^2 - \alpha_2^2 y_2^2 - \alpha_3^2 y_3^2 + y_2 y_3 (1 - \alpha_1^2 + \alpha_2^2 + \alpha_3^2), \\ \dot{y}_2 &= -2(y_1 - y_3) y_2, \quad \dot{y}_3 = -2(y_1 - y_2) y_3, \end{aligned}$$

with cofactor $K = (a_1 + a_2 + a_3) y_1 - a_2 y_2 - a_3 y_3$. Hence, it satisfies

$$\begin{aligned} &-2(y_1 - y_3) y_2 b_2 - 2(y_1 - y_2) y_3 b_3 + b_1(-y_1^2 - \alpha_2^2 y_2^2 - \alpha_3^2 y_3^2 + y_2 y_3 (1 - \alpha_1^2)) \\ &\quad + b_1 y_2 y_3 (\alpha_2^2 + \alpha_3^2) = ((a_1 + a_2 + a_3) y_1 - a_2 y_2 - a_3 y_3)(b_1 y_1 + b_2 y_2 + b_3 y_3). \end{aligned} \tag{21}$$

Equating the terms in (21) of degree two in the variable $y_1$, we get that $-b_1 = (a_1 + a_2 + a_3) b_1$ and thus we can have two different cases.

*Case 1:* $b_1 \neq 0$. In this case,

$$a_1 + a_2 + a_3 = -1. \tag{22}$$

Then, equating in (21) the terms of degree two in the variables $y_2$ and $y_3$, we get

$$\alpha_2^2 b_1 = a_2 b_2, \quad \alpha_3^2 b_1 = a_3 b_3. \tag{23}$$

Now, equating in (21) the coefficients of the variables $y_1 y_2$ and $y_1 y_3$, we obtain

$$b_2 = a_2 b_1, \quad b_3 = a_3 b_1. \tag{24}$$

Then, from (23) and (24) it follows that $a_2^2 = \alpha_2^2$ and $a_3^2 = \alpha_3^2$. Finally, equating in (21) the coefficients of the variable $y_2 y_3$, and using (23), (24) and (22), we get that

$$\begin{aligned} 0 &= 2b_2 + 2b_3 + (1 - \alpha_1^2 + a_2^2 + a_3^2) b_1 + a_3 b_2 + a_2 b_3 \\ &= b_1((a_2 + a_3)^2 + 2(a_2 + a_3) + 1 - \alpha_1^2) \\ &= b_1((1 + a_1)^2 - 2(1 + a_1) + 1 - \alpha_1^2) = b_1(a_1^2 - \alpha_1^2), \end{aligned}$$

which yields $a_1^2 = \alpha_1^2$. Thus, $a_1 = \pm\alpha_1$, $a_2 = \pm\alpha_2$, $a_3 = \pm\alpha_3$, and from (22) we obtain $\pm\alpha_1 \pm \alpha_2 \pm \alpha_3 = -1$, a contradiction with the fact that $(\alpha_1, \alpha_2, \alpha_3) \in \Gamma$. Thus, this case is not possible.

*Case 2:* $b_1 = 0$. In this case, equating the terms of degree two in the variables $y_2$ and $y_3$, we get $a_2 b_2 = 0$ and $a_3 b_3 = 0$. Clearly, $b_2 = b_3 = 0$ would imply that $g = 0$, and thus $f = 0$, a contradiction. Hence this case is not possible. We consider the three different possibilities.

*Case 2.1:* $b_1 = a_2 = b_3 = 0$, $b_2 \neq 0$. In this case, $g = b_2 y_2$ and, after dividing by $b_2$, from (21) we get $-2(y_1 - y_3) y_2 = (a_1 + a_3) y_2 y_1 - a_3 y_2 y_3$, which yields $a_3 = -2$, $a_1 = 0$. Then, from (20) we get $f = b_2(x_1 - x_2) = b_2 H_1$ and $K = -2x_3$.

*Case 2.2:* $b_1 = a_3 = b_2 = 0$, $b_3 \neq 0$. In this case, $g = b_3 y_3$ and, after dividing by $b_3$, from (21) we get $-2(y_1 - y_2) y_3 = (a_1 + a_2) y_3 y_1 - a_2 y_2 y_3$, which yields $a_2 = -2$, $a_1 = 0$. Then, from (20) we get $f = b_3(x_1 - x_3) = b_3 H_2$ and $K = -2x_2$.

*Case 2.3:* $b_1 = a_2 = a_3 = 0$. In this case, $g = b_2 y_2 + b_3 y_3$ and from (21) we get $-2(y_1 - y_3) y_2 b_2 - 2(y_1 - y_2) y_3 b_3 = a_1 b_2 y_2 y_1 + a_1 b_3 y_1 y_3$, which yields $b_2 = -b_3$ and $a_1 = -2$. Then, from (20) we get $f = b_3(x_2 - x_3) = b_3 H_3$ and $K = -2x_1$.

Thus, the proposition is proved. □

**Proposition 16.** *For $(\alpha_1, \alpha_2, \alpha_3) \in \Gamma$, let $f$ be an irreducible homogeneous Darboux polynomial of system (1) with degree at least two and cofactor $K$, as in Proposition 13. Then $K = 0$.*

To prove Proposition 16, we will show that each of the coefficients $a_1$, $a_2$ and $a_3$ in the definition of $K$ given in Proposition 13 is zero for any Darboux polynomial of system (1) of degree greater than or equal to two. For this, we shall need the following preliminary result, which describes the Darboux polynomials and their cofactors of system (1) restricted to each one of $H_j$, $j = 1, 2, 3$, defined in (2).

**Proposition 17.** *Let $\overline{f} := \overline{f}(x_2, x_3)$ be a homogeneous Darboux polynomial of degree $n \geq 2$ of system (1) restricted to $x_1 = x_2$ with cofactor $K = c_2x_2 + c_3x_3$, where $(c_2, c_3) \in \mathbb{C}^2 \setminus \{(0, 0)\}$. The following holds:*
*If $\alpha_3 = 0$ and $\overline{f} \neq 0$, then $c_3 = 0$, $c_2 \in \mathbb{N}^- \setminus \{0\}$, $-2n \leq c_2 \leq -n$ and there exists a positive constant $d_0$ such that $\overline{f} = d_0(x_2 - x_3)^{-n-c_2}x_2^{c_2+2n}$.*
*If $\alpha_3 \neq 0$ and $\overline{f} \neq 0$, then $c_3/\alpha_3 \in \mathbb{Z}$, $c_2+c_3 \in \mathbb{N}^- \setminus \{0\}$, $-2n \leq c_2+c_3 \leq -n$ and $-n/2 \leq c_3/\alpha_3 \leq n/2$. Moreover, there exists a positive constant $d_1$ such that $\overline{f} = d_1(x_2 - x_3)^{-n-c_2-c_3}(x_2 - \alpha_3(x_2 - x_3))^{A_1}(x_2 + \alpha_3(x_2 - x_3))^{B_1}$, where*

$$A_1 = \frac{1}{2}\left(c_2 + c_3 + 2n - \frac{c_3}{\alpha_3}\right), \quad B_1 = \frac{1}{2}\left(c_2 + c_3 + 2n + \frac{c_3}{\alpha_3}\right). \tag{25}$$

**Proof.** Let $\overline{f}$ be a homogeneous Darboux polynomial of system (1) restricted to $x_1 = x_2$ with cofactor $K = c_2x_2 + c_3x_3$, where $(c_2, c_3) \in \mathbb{C}^2 \setminus \{(0, 0)\}$. We introduce the change of variables in (6) and denote $g_1(y_2, z_2) = \overline{f}(x_2, x_3)$. Then, it is clear that $g_1$ satisfies

$$-(y_2^2 + \alpha_3^2 z_2^2)\frac{\partial g_1}{\partial y_2} - 2y_2z_2\frac{\partial g_1}{\partial z_2} = ((c_2 + c_3)y_2 - c_3z_2)g_1. \tag{26}$$

Now, we introduce the change

$$u_2 = y_2/z_2, \quad w_2 = z_2. \tag{27}$$

In the new variables, we have that $g_1(y_2, z_2) = h(u_2, w_2)$ with $h$ a polynomial. We consider the polynomial $h$ in the region $w_2 \neq 0$ and use its finite power series to define a polynomial $h_1 := h_1(u_2, w_2)$ for arbitrary $w_2$. Furthermore, if $g_1 = \sum_{k=0}^{n} g_{k,n-k}y_2^k z_2^{n-k}$, then $h_1 = w_2^n \sum_{k=0}^{n} g_{k,n-k}u_2^k := w_2^n F(u_2)$. Clearly, in view of (26), $F := F(u_2)$ satisfies

$$(u_2^2 - \alpha_3^2)\frac{\mathrm{d}F}{\mathrm{d}u_2} = ((c_2 + c_3 + 2n)u_2 - c_3)F. \tag{28}$$

Now, we consider two different cases.

*Case 1:* $\alpha_3 = 0$. In this case, solving (28), we obtain that there exists a constant $d_0 > 0$ such that $F = d_0u_2^{c_2+c_3+2n}e^{\frac{c_3}{u_2}}$. Since $F$ must be a polynomial, it must hold that $c_3 = 0$ and $c_2 \in \mathbb{Z}$ with $c_2 \geq -2n$. Hence, in this case

$$h_1 = d_0w_2^n u_2^{c_2+2n}. \tag{29}$$

In any other case, $F = 0$, which obviously yields $h_1 = 0$ and thus $\overline{f} = 0$. When $c_3 = 0$, $c_2 \in \mathbb{Z}$, $c_2 \geq -2n$, from (29) and (27), $g_1 = d_0z_2^{-n-c_2}y_2^{c_2+2n}$. Since $g_1$ must be a polynomial, it must hold that $-n-c_2 \geq 0$ and, thus, $c_2 \in \mathbb{N}^- \setminus \{0\}$ with $-2n \leq c_2 \leq -n$. Hence, in this case, the proposition follows, taking into account (6).

*Case 2:* $\alpha_3 \neq 0$. In this case, solving (28), we obtain that there exists a constant $d_1$ such that $F = d_1(u_2 - \alpha_3)^{A_1}(u_2 + \alpha_3)^{B_1}$, where $A_1$ and $B_1$ were introduced in (25). Since $F$ must be a polynomial, $A_1, B_1 \in \mathbb{N}$. In particular, since $B_1 - A_1 = c_3/\alpha_3$, we get that

$$c_3/\alpha_3 \in \mathbb{Z} \quad \text{with} \quad -\frac{c_2 + c_3 + 2n}{2} \leq \frac{c_3}{\alpha_3} \leq \frac{c_2 + c_3 + 2n}{2}. \tag{30}$$

Furthermore, since $A + B \in \mathbb{N}$, we obtain that $c_2 + c_3 \in \mathbb{Z}$ and $c_2 + c_3 \geq -2n$. Hence, in this case

$$h_1 = d_1w_2^n(u_2 - \alpha_3)^{A_1}(u_2 + \alpha_3)^{B_1}, \tag{31}$$

and in any other case $F = 0$, which obviously yields $h_1 = 0$, and thus $\overline{f} = 0$.

When $c_3/\alpha_3 \in \mathbb{Z}$, $c_2 + c_3 \in \mathbb{Z}$, $c_2 + c_3 \geq -2n$, from (31) and (27) we get that

$$g_1 = d_1 z_2^{-n-c_2-c_3}(y_2 - \alpha_3 z_2)^{A_1}(y_2 + \alpha_3 z_2)^{B_1}. \tag{32}$$

Since $g_1$ must be a polynomial, it must hold that $-n - c_2 - c_3 \geq 0$ and thus, $c_2 + c_3 \in \mathbb{N}^- \setminus \{0\}$ with $-2n \leq c_2 + c_3 \leq -n$. Then, from (30), $-n/2 \leq c_3/\alpha_3 \leq n/2$. Therefore, the proposition follows from (32) and (6). □

The following two propositions can be proved in a similar manner to Proposition 17. Hence, their proofs have been omitted.

**Proposition 18.** *Let $\hat{f} := \hat{f}(x_2, x_3)$ be a homogeneous Darboux polynomial of degree $n \geq 2$ of system (1) restricted to $x_1 = x_3$ with cofactor $K = c_2 x_2 + c_3 x_3$, where $(c_2, c_3) \in \mathbb{C}^2 \setminus \{(0, 0)\}$. Then, it holds that:*
*If $\alpha_2 = 0$ and $\hat{f} \neq 0$, then $c_2 = 0$, $c_3 \in \mathbb{N}^- \setminus \{0\}$, $-2n \leq c_3 \leq -n$ and there exists a positive constant $d_2$ such that $\overline{f} = d_2(x_3 - x_2)^{-n-c_3} x_3^{c_3+2n}$.*
*If $\alpha_2 \neq 0$ and $\hat{f} \neq 0$, then $c_2/\alpha_2 \in \mathbb{Z}$, $c_2 + c_3 \in \mathbb{N}^- \setminus \{0\}$, $-2n \leq c_2 + c_3 \leq -n$, $-n/2 \leq c_2/\alpha_2 \leq n/2$, and there exists a positive constant $d_3$ such that $\hat{f} = d_3(x_3 - x_2)^{-n-c_2-c_3}(x_3 - \alpha_2(x_3 - x_2))^{A_2}(x_3 + \alpha_2(x_3 - x_2))^{B_2}$, where $A_2 = \frac{1}{2}\left(c_2 + c_3 + 2n - \frac{c_2}{\alpha_2}\right)$ and $B_2 = \frac{1}{2}\left(c_2 + c_3 + 2n + \frac{c_2}{\alpha_2}\right)$.*

**Proposition 19.** *Let $\tilde{f} := \tilde{f}(x_1, x_3)$ be a homogeneous Darboux polynomial of degree $n \geq 2$ of system (1) restricted to $x_2 = x_3$ with cofactor $K = c_1 x_1 + c_3 x_3$, where $(c_1, c_3) \in \mathbb{C}^2 \setminus \{(0, 0)\}$. Then, it holds that:*
*If $\alpha_1 = 0$ and $\tilde{f} \neq 0$, then $c_1 = 0$, $c_3 \in \mathbb{N}^- \setminus \{0\}$, $-2n \leq c_3 \leq -n$ and there exists a positive constant $d_4$ such that $\overline{f} = d_4(x_1 - x_3)^{-n-c_3} x_1^{c_3+2n}$.*
*If $\alpha_1 \neq 0$ and $\tilde{f} \neq 0$, then $c_1/\alpha_1 \in \mathbb{Z}$, $c_1 + c_3 \in \mathbb{N}^- \setminus \{0\}$, $-2n \leq c_1 + c_3 \leq -n$, $-n/2 \leq c_1/\alpha_1 \leq n/2$, and there exists a positive constant $d_5$ such that $\tilde{f} = d_5(x_1 - x_3)^{-n-c_1-c_3}(x_1 - \alpha_1(x_1 - x_3))^{A_3}(x_1 + \alpha_1(x_1 - x_3))^{B_3}$, where $A_3 = \frac{1}{2}\left(c_1 + c_3 + 2n - \frac{c_1}{\alpha_1}\right)$ and $B_3 = \frac{1}{2}\left(c_1 + c_3 + 2n + \frac{c_1}{\alpha_1}\right)$.*

**Proof of Proposition 16.** The proof will be done by reduction to the absurd. Let $(\alpha_1, \alpha_2, \alpha_3) \in \Gamma$ and $f$ be an irreducible homogeneous Darboux polynomial of degree $n \geq 2$ of system (1) with cofactor $K = a_1 x_1 + a_2 x_2 + a_3 x_3$ with $(a_1, a_2, a_3) \in \mathbb{C}^3 \setminus \{(0, 0, 0)\}$. From the fact that $f$ is irreducible, it is clear that $f_1 \neq 0$, $f_2 \neq 0$ and $f_3 \neq 0$, since otherwise $f$ would be divisible by $x_1 - x_2$ or by $x_1 - x_3$ or by $x_2 - x_3$, a contradiction.

Furthermore, since $f_1$, $f_2$ and $f_3$ are, respectively, homogeneous Darboux polynomials of system (1) restricted to $x_1 = x_2$, $x_1 = x_3$ and $x_2 = x_3$, we can apply Proposition 17 with $\overline{f} = f_1$, $c_2 = a_1 + a_2$, $c_3 = a_3$, Proposition 18 with $\hat{f} = f_2$, $c_2 = a_2$, $c_3 = a_1 + a_3$, and Proposition 19 with $\tilde{f} = f_3$, $c_1 = a_1$ and $c_3 = a_2 + a_3$. Doing so, we consider different cases.

*Case 1:* $\alpha_1 = \alpha_2 = \alpha_3 = 0$. In this case, from Propositions 17–19, we have $a_1 = a_2 = a_3 = 0$, a contradiction with the fact that $f$ is an homogeneous Darboux polynomial with non-zero cofactor $K$.

*Case 2: There exists $\{i, j, k\} \in \{1, 2, 3\}$ with $i \neq j$, $i \neq k$, $j \neq k$ such that $\alpha_i = \alpha_j = 0$ and $\alpha_k \neq 0$.* Without loss of generality, we can assume $i = 1$, $j = 2$ and $k = 3$. In this case, from Propositions 18 and 19, we have $a_1 = a_2 = 0$ and $a_3 \in \mathbb{N}^- \setminus \{0\}$. Furthermore, from Proposition 17, we have $a_3/\alpha_3 \in \mathbb{Z} \setminus \{0\}$. Then, there exists $N_1 \in \mathbb{Z} \setminus \{0\}$ such that $\alpha_3 N_1 = a_3 \in \mathbb{N}^- \setminus \{0\}$, a contradiction with the fact that $(\alpha_1, \alpha_2, \alpha_3) \in \Gamma$.

*Case 3: There exists $\{i, j, k\} \in \{1, 2, 3\}$ with $i \neq j$, $i \neq k$, $j \neq k$ such that $\alpha_i = 0$, $\alpha_j \neq 0$ and $\alpha_k \neq 0$.* Without loss of generality, we can assume $i = 1$, $j = 2$ and $k = 3$. Then, from Proposition 19 we have $a_1 = 0$, and from Propositions 18 and 17 we have that $a_2/\alpha_2 \in \mathbb{Z}$ and $a_3/\alpha_3 \in \mathbb{Z}$ with $a_2 + a_3 \in \mathbb{N}^- \setminus \{0\}$. Then, since either $a_2 \neq 0$ or $a_3 \neq 0$, there exists $N_1, N_2 \in \mathbb{Z}$ with $N_1 + N_2 \neq 0$ such that $\alpha_2 N_1 + \alpha_3 N_2 = a_2 + a_3 \in \mathbb{N}^- \setminus \{0\}$, a contradiction with the fact that $(\alpha_1, \alpha_2, \alpha_3) \in \Gamma$.

*Case 4: $\alpha_1 \neq 0$, $\alpha_2 \neq 0$ and $\alpha_3 \neq 0$.* In this case, from Propositions 17–19, we have $a_1/\alpha_1 \in \mathbb{Z}$, $a_2/\alpha_2 \in \mathbb{Z}$, $a_3/\alpha_3 \in \mathbb{Z}$ and $a_1 + a_2 + a_3 \in \mathbb{N}^- \setminus \{0\}$. Then, since either $a_1 \neq 0$ or $a_2 \neq 0$ or $a_3 \neq 0$, there exists $N_1, N_2, N_3 \in \mathbb{Z}$ with $N_1 + N_2 + N_3 \neq 0$ such that $\alpha_1 N_1 + \alpha_2 N_2 + \alpha_3 N_3 = a_1 + a_2 + a_3 \in \mathbb{N}^- \setminus \{0\}$, a contradiction with the fact that $(\alpha_1, \alpha_2, \alpha_3) \in \Gamma$.

Thus, the proposition is proved. □

**Proof of Theorem 12.** If $f$ has degree one, the proof of Theorem 12 follows directly from Proposition 15. Now, assume that system (1) has a irreducible Darboux polynomial with degree at least two and with cofactor $K \neq 0$ given as in Proposition 13. From Proposition 14, we can assume that $f$ is an homogeneous irreducible Darboux polynomial of degree at least two and cofactor $K \neq 0$. Then, from Proposition 16, we reach a contradiction. Thus, all irreducible Darboux polynomials with cofactor $K \neq 0$, as in Proposition 13, must come from irreducible Darboux polynomials of system (1) of degree one, i.e., $H_1 = x_1 - x_2$, $H_2 = x_1 - x_3$ and $H_3 = x_2 - x_3$. Furthermore, from Theorem 2, we know that all Darboux polynomials of system (1) with cofactor zero, i.e., polynomial first integrals, are constants. These facts, together with Lemma 6, imply the proof of the theorem. □

**Proof of Theorem 4.** By Theorem 12, it follows that every Darboux polynomial of system (1) is of the form $cH_1^m H_2^n H_3^l$ with cofactor

$$K = -2(mx_3 + nx_2 + lx_1), \tag{33}$$

where $m, n$ and $l$ are non-negative integers, and $c$ is some constant.

From Proposition 7 and Theorem 2, the existence of a non-constant rational first integral implies the existence of two coprime Darboux polynomials with the same non-zero cofactor. So, the first integral must be of the form $R/S = c_0 H_1^{m_1} H_2^{n_1} H_3^{l_1} / c_1 (H_1^{m_2} H_2^{n_2} H_3^{l_2})$ with at most one $m_i$, $n_i$ and $l_i$ non-zero, and the cofactors of $R$ and $S$ must be equal.

Then, according to (33), the equality of the cofactors of $R$ and $S$ implies that $2(m_1 - m_2)x_3 + 2(n_1 - n_2)x_2 + 2(l_1 - l_2)x_1 = 0$. Hence, $m_1 = m_2$, $n_1 = n_2$, $l_1 = l_2$, a contradiction with the fact that $R$ and $S$ are coprime. Thus, the theorem is proved. □

## 5. Proof of Theorem 5

We recall that the equation defining the exponential factor $F = \exp(h/g)$ with cofactor $L$ for system (1) is

$$\dot{x}_1 \frac{\partial}{\partial x_1}\left(\frac{h}{g}\right) + \dot{x}_2 \frac{\partial}{\partial x_2}\left(\frac{h}{g}\right) + \dot{x}_3 \frac{\partial}{\partial x_3}\left(\frac{h}{g}\right) = L, \tag{34}$$

where we have simplified the common factor $F$, and

$$L = b_0 + b_1 x_1 + b_2 x_2 + b_3 x_3. \tag{35}$$

According to Proposition 8 and Theorems 2 and 12, if system (1) has exponential factors, they must be of the form $\exp(h/(H_1^{n_1} H_2^{n_2} H_3^{n_3}))$, where $h \in \mathbb{C}[x_1, x_2, x_3]$ and $n_1, n_2, n_3 \in \mathbb{N} \cup \{0\}$.

To prove Theorem 5 with the help of Theorem 9 we shall introduce and prove the following statement.

**Proposition 20.** *For $(\alpha_1, \alpha_2, \alpha_3) \in \Gamma$, system (1) does not admit non-constant exponential factors.*

**Proof.** We start by showing that if system (1) has an exponential factor of the form $\exp(h)$, then $h$ is a constant. Applying (34) with $h/g = h$, we get

$$\dot{x}_1 \frac{\partial h}{\partial x_1} + \dot{x}_2 \frac{\partial h}{\partial x_2} + \dot{x}_3 \frac{\partial h}{\partial x_3} = L \tag{36}$$

with $L$ given by (35). Taking $x_1 = x_2 = x_3 = 0$ in (36), we obtain $b_0 = 0$. Now, letting $x_1 = x_2 = 0$ in (36), we get

$$-\alpha_3^2 x_3^2 \left( \frac{\partial f}{\partial x_1}_{|x_1=x_2=0} + \frac{\partial f}{\partial x_2}_{|x_1=x_2=0} + \frac{\partial f}{\partial x_3}_{|x_1=x_2=0} \right) = b_3 x_3, \tag{37}$$

where, for $j = 1, 2, 3$, $\frac{\partial f}{\partial x_j}_{|x_1=x_2}$ means the restriction to $x_1 = x_2$ of $\partial f/\partial x_j$. Eq. (37) obviously implies $b_3 = 0$. Analogously, setting $x_1 = x_3 = 0$ in (36) we get $b_2 = 0$, and setting $x_2 = x_3 = 0$ in (36) we get $b_1 = 0$. Thus, $L = 0$ and (36) reduces to $\frac{dh}{dt} = 0$, i.e., $h$ is a polynomial first integral of system (1). From Theorem 2, we get that $h$ is a constant.

Suppose that $\exp(h/(H_1^{n_1}H_2^{n_2}H_3^{n_3}))$ is an exponential factor of system (1), where $n_1, n_2, n_3$ are non-negative integers with at least one of them positive, and $h$ is coprime with $H_1$, $H_2$ and $H_3$. Then, $h$ satisfies

$$\dot{x}_1\frac{\partial h}{\partial x_1} + \dot{x}_2\frac{\partial h}{\partial y_2} + \dot{x}_3\frac{\partial h}{\partial x_3} - \left(\frac{\dot{H}_1}{H_1}n_1 + \frac{\dot{H}_2}{H_2}n_2 + \frac{\dot{H}_3}{H_3}n_3\right)h = LH_1^{n_1}H_2^{n_2}H_3^{n_3}. \tag{38}$$

Without loss of generality, we can assume that $n_1 > 0$. Taking $H_1 = 0$ in (38) and denoting by $h_1$ the restriction of $h$ to $H_1 = 0$, we conclude that $h_1$ satisfies

$$-(x_2^2 + \alpha_3^2(x_2 - x_3)^2)\frac{\partial h_1}{\partial x_2} + (x_2^2 - 2x_2x_3 - \alpha_3^2(x_2 - x_3)^2)\frac{\partial h_1}{\partial x_3} = -2(n_1x_3 + (n_2 + n_3)x_2)h_1. \tag{39}$$

Since, by hypothesis, $h$ is coprime with $H_1$, we have that $h_1 \neq 0$. Furthermore, from (39), $h_1$ is a Darboux polynomial of system (1) restricted to $x_1 = x_2$ with cofactor $K = -2(n_1x_3 + (n_2 + n_3)x_2)$. In view of Proposition 14, we can assume that $h_1$ is homogeneous. Then, from Proposition 17 and since $h_1 \neq 0$, we must have:
If $\alpha_1 = 0$, then $-2n_1 = 0$, a contradiction with the fact that $n_1 > 0$.
If $\alpha_1 \neq 0$, then $-2n_1/\alpha_1 \in \mathbb{Z} \setminus \{0\}$. Hence, there exists $N \in \mathbb{Z}^- \setminus \{0\}$ such that $N\alpha_1 = -2n_1 \in \mathbb{N}^- \setminus \{0\}$, a contradiction with the fact that $(\alpha_1, \alpha_2, \alpha_3) \in \Gamma$.

This completes the proof of the proposition. □

**Proof of Theorem 5.** From Theorems 2, 9 and 12 and Proposition 20, if system (1) has a Darboux first integral $G$, then $G = cH_1^{\lambda_1}H_2^{\lambda_2}H_3^{\lambda_3}$ where $c, \lambda_1\lambda_2, \lambda_3 \in \mathbb{C}$. Since $G$ is a first integral, it must hold $2(\lambda_1x_3 + \lambda_2x_2 + \lambda_3x_1) = 0$. This implies that $\lambda_1 = \lambda_2 = \lambda_3 = 0$, and completes the proof of the theorem. □


## Acknowledgement

The author was supported by the Center for Mathematical Analysis, Geometry, and Dynamical Systems, and through Fundação para a Ciência e a Tecnologia by Program POCTI/FEDER, Program POSI, and the grant SFRH/BPD/14404/2003.


## References


[1] M. Ablowitz, J. Chakravarty, R. Halburd, The generalized Chazy equation and Schwarzian triangle functions, Asian J. Math. 2 (1998) 1–6.
[2] J.D. Barrow, Chaotic behaviour in general relativity, Phys. Rep. 85 (1982) 1–49.
[3] A. Burd, N. Buric, G.F.R. Ellis, A numerical analysis of chaotic behaviour in Bianchi IX models, Gen. Relativity Gravitation 22 (1990) 349–363.
[4] L. Cairo, J. Llibre, Darboux integrability for 3D Lotka-Volterra systems, J. Phys. A 33 (2000) 2395–2406.
[5] S. Chakravarty, M. Ablowitz, Integrability, monodromy, evolving deformations and self-dual Bianchi IX systems, Phys. Rev. Lett. 76 (1996) 857–860.
[6] S. Chakravarty, M. Ablowitz, L. Takhtajan, Self dual Yang-Mills equation and new special functions in integrable systems, in: Nonlinear Evolution Equations and Dynamical System, 1992, pp. 3–11.
[7] S. Chakravarty, R. Halburd, First integrals of a generalized Darboux–Halphen system, J. Math. Phys. 44 (2003) 1751–1762.
[8] J. Chavarriga, H. Giacomini, J. Giné, J. Llibre, Darboux integrability and the inverse integrating factor, J. Differential Equations 194 (2003) 116–139.
[9] J. Chavarriga, J. Llibre, Invariant algebraic curves and rational first integrals for planar polynomial vector fields, J. Differential Equations 169 (2001) 1–16.
[10] C. Christopher, Invariant algebraic curves and conditions for a center, Proc. Roy. Soc. Edinburgh 124A (1994) 1209–1229.
[11] C. Christopher, J. Llibre, Integrability via invariant algebraic curves for planar polynomial differential systems, Ann. Differential Equations 16 (2000) 5–19.
[12] G. Darboux, Sur la théorie des coordonnées curvilignes et les systémes orthogonaux, Ann. Ec. Normale Supér 7 (1878) 101–150.
[13] G. Darboux, Mèmoire sur les équations différentielles algébraiques du premier ordre et du premier degré (Métanges), Bull. Sci. Math. 2éme série 2 (1878) 60–96, 123–144, 151–200.
[14] J. Demaret, Y. de Rop, The fractal nature of the power spectrum as an indicator of chaos in the Bianchi IX cosmological model, Phys. Lett. B 299 (1993) 223–228.
[15] R. Halburd, Solvable models of relativistic charged spherically symmetric fluids, Classical Quantum Gravity 18 (2001) 11–25.
[16] G. Halphen, Sur un système d'équations différetielles, C. R. Acad. Sci. Paris 92 (1881) 1101–1103.
[17] N. Hitchin, Hipercomplex manifolds and the space of framings, in: The Geometric Univers. Proccedings, Oxford, 1996, pp. 9–30.
[18] D. Hobill, D. Bernstein, M. Welge, D. Simkings, The mixmaster cosmology as a dynamical system, Classical Quantum Gravity 8 (1991) 1155–1171.

[19] J. Llibre, X. Zhang, Darboux integrability of real polynomial vector fields on regular algebraic hypersurfaces, Rend. Circ. Mat. Palermo 51 (2002) 109–126.
[20] Y. Ohyama, Systems of nonlinear differential equations related to second order linear equations, Osaka J. Math. 33 (1996) 927–949.
[21] S.E. Rugh, B.J.T. Jones, Chaotic behaviour and oscillating three-volumes in Bianchi IX universes, Phys. Lett. A 147 (1990) 353–359.
[22] M. Szydlowski, M. Biesiada, Chaos in mixmaster models, Phys. Rev. D 44 (1991) 2369–2374.
[23] L. Takhtajan, On foundation of the generalized Nambu mechanics, Commun. Math. Phys. 160xi (1994) 295–315.
[24] C. Valls, Analytic first integrals of the Halphen system, J. Geom. Phys (in press).
[25] P. Vassiliou, Vessiot structure for manifolds of $(p, q)$-hyperbolic type: Darboux integrability and symmetry, Trans. Amer. Math. Soc. 353 (2001) 1705–1739.
[26] X. Zhang, Invariant hyperplanes and Darboux integrability of polynomial vector fields, J. Phys. A 15 (1999) 1621–1637.